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VERTICES AND THE CJT EFFECTIVE POTENTIAL

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ABSTRACT

The Cornwall-Jackiw-Tomboulis effective potential is modified to include a functional dependence on the fermion-gauge particle vertex, and applied to a quark confining model of chiral symmetry breaking.

1. Introduction

The study of dynamical chiral symmetry breaking has benefited greatly from the Cornwall-Jackiw-Tomboulis (CJT) effective potential [1]. The CJT potential is an effective potential of composite operators, specifically the propagators of a theory. Minimizing the CJT potential with respect to a propagator gives the Schwinger-Dyson (SD) equation for that propagator. The SD equation can have more than one solution. These solutions must be inserted in the CJT potential to find the solution corresponding to a stable minimum in the CJT potential, thus determining the physical solution. In its simplest form the CJT potential as a functional of propagators is of practical use only in solving propagator SD equations, which have a severe truncation of their vertex SD equations, as is the case for the ladder approximation, where the full vertex is replaced by the bare vertex. The usual CJT potential does not have a functional dependence on the vertices of a theory, as a result the SD equations of the vertices cannot be found by minimizing the CJT potential. There exist aternative means of constructing full vertices, such as solving the Ward identity of the vertex (the gauge technique), to arrive at a full vertex as a functional of the propagators, which is valid in the infrared regime. Putting the gauge technique vertex in the propagtor SD equation by hand is inconsistent with minimizing the CJT potential, which is not a functional of the gauge technique vertex. Therefore testing the stability of the CJT potential with solutions found from propagator SD equations containing the gauge technique vertex can be unreliable (as will be demonstrated in section 3).

We will extend the CJT formalism to write an effective potential for QED which has a functional dependence on the fermion-fermion-gauge particle vertex in addition to the propagators, and apply it to a confining model of dynamical chiral symmetry breaking. Interestingly, it will be found that the chiral symmetry breaking solution of the effective quark propagator SD equation minimizes the modified CJT potential. Section 2 contains a derivation of the modified CJT potential for QED, while section 3 applies the results to a specific model.

2. CJT Potential as a Function of the Vertex

Consider massless QED with its gauge particle and fermion propagators, and fermion-fermion-gauge particle vertex denoted as $\Delta_{\mu\nu}(x,y)$, $S(x,y)$, and $\Gamma_\mu(x,y,z)$ respectively. The CJT effective action is given by

$$\Gamma(\Delta_{\mu\nu}, S) = \Gamma_0(\Delta_{\mu\nu}, S) + \Gamma_2(\Delta_{\mu\nu}, S), \tag{1}$$

where

$$\begin{aligned}\Gamma_0(\Delta_{\mu\nu}, S) = \; & \tfrac{-i\hbar}{2} Tr\ ln({\Delta_{0\mu\alpha}}^{-1}\Delta_{\alpha\nu}) + \tfrac{i\hbar}{2} Tr({\Delta_{0\mu\alpha}}^{-1}\Delta_{\alpha\nu} - g_{\mu\nu}) \\ & + i\hbar Tr\ ln(S_0^{-1}S) - i\hbar Tr(S_0^{-1}S - 1),\end{aligned} \tag{2}$$

while $\Gamma_2(\Delta_{\mu\nu}, S)$ is the infinite sum over all possible 2 point-irreducible (2PI) Feynman graphs involving $\Delta_{\mu\nu}$ and S, and $\Delta_{0\mu\nu}$ and S_0 are the bare gauge particle and fermion propagators respectively. Including a vertex source in the generator of connected Green's function, W, gives $W = W(G_{\mu\nu}, K, \rho_\mu)$, where $G_{\mu\nu}$, K, and ρ_μ are the sources for $\Delta_{\mu\nu}$, S, and Γ_μ respectively. Perfoming a triple Legendre transformation, we find

$$\begin{aligned}\Gamma(\Delta_{\mu\nu}, S, \Gamma_\mu) = \; & W - i\hbar \int d^4x d^4y \left(\tfrac{1}{2} G_{\mu\nu}(x,y)\Delta^{\mu\nu}(y,x) - K(x,y)S(y,x)\right) \\ & - i\hbar \int d^4x d^4y d^4z \rho^\mu(x,y,z)\Gamma_\mu(z,y,x),\end{aligned} \tag{3}$$

corresponding to

$$\Gamma(\Delta_{\mu\nu}, S, \Gamma_\mu) = \Gamma_0(\Delta_{\mu\nu}, S) + \Gamma_2(\Delta_{\mu\nu}, S, \Gamma_\mu), \tag{4}$$

where $\Gamma_2(\Delta_{\mu\nu}, S, \Gamma_\mu)$ is an infinite sum over all 2PI Feynman graphs containing all possible interactions of full propagators and full vertices, as well as all the Feynman graphs that make up $\Gamma_2(\Delta_{\mu\nu}, S)$. Evaluating the effective action at a stationary point gives

$$\frac{\delta\Gamma}{\delta\Delta_{\mu\nu}} = -\frac{i\hbar}{2} G_{\mu\nu}|_{G_{\mu\nu}=0}, \tag{5}$$

$$\frac{\delta\Gamma}{\delta S} = i\hbar K|_{S=0}, \tag{6}$$

and

$$\frac{\delta\Gamma}{\delta\Gamma_\mu} = -i\hbar\rho_\mu|_{\rho_\mu=0}, \tag{7}$$

where Eq.'s 5-7 are the SD equations for $\Delta_{\mu\nu}(x-y)$, $S(x-y)$, and $\Gamma_\mu(x-y, y-z)$ respectively, where the propagators(vertex) are(is) a funtion of 1(2) variables, rather than 2(3) before the sources were set to zero. At the stationary point the effective action can be written as

$$\Gamma(\Delta_{\mu\nu}, S, \Gamma_\mu) = -V(\Delta_{\mu\nu}, S, \Gamma_\mu)\int d^4x, \tag{8}$$

where the effective potential, $V(\Delta_{\mu\nu}, S, \Gamma_\mu)$, is given by [2]

$$V(\Delta_{\mu\nu}, S, \Gamma_\mu) = V_0(\Delta_{\mu\nu}, S, \Gamma_\mu) + V_2(\Delta_{\mu\nu}, S, \Gamma_\mu), \tag{9}$$

$$\begin{aligned} V_0(\Delta_{\mu\nu}, S, \Gamma_\mu) = \quad & -i\hbar \int \tfrac{d^4p}{2\pi^4} Tr\Big(-\tfrac{1}{2}ln({\Delta_{0\mu\alpha}}^{-1}(p)\Delta_{\alpha\nu}(p)) \\ & +\tfrac{1}{2}(\Delta_{0\mu\alpha}(p)^{-1}\Delta_{\alpha\nu}(p) - g_{\mu\nu}) \\ & +ln(S_0(p)^{-1}S(p)) - (S_0(p)^{-1}S(p) - 1)\Big), \end{aligned} \tag{10}$$

where $V_2(\Delta_{\mu\nu}, S, \Gamma_\mu)$ is the infinite sum over vacuum graphs shown in Fig. 1. Eqs. 5-7 can alternatively be written as

$$\frac{\delta V}{\delta\Delta_{\mu\nu}(p)} = 0, \tag{11}$$

$$\frac{\delta V}{\delta S(p)} = 0, \tag{12}$$

and

$$\frac{\delta V}{\delta\Gamma_\mu(p,q)} = 0, \tag{13}$$

which are displayed in Figs. 2-4 respectively. The coffecients of the graphs on the right hand side of Fig. 1 are determined by first examining the vertex SD

equation shown in Fig. 4, and then either of the propagator SD equations (Figs. 2 and 3); using Fig. 4 it can be seen that $V_2(S, \Gamma_\mu) = V_2(S)$ as we expect.

The vertex SD equation is not solvable as the infinite series shown in Fig. 4. Truncating the series beyond the first graph on the right hand side of Fig. 4, and keeping only the first graph on the right hand side of Fig. 2 (choosing the Landau gauge) is a common approximation (ladder approximation) used to solve the fermion SD equation. Another approximation scheme of determining the vertex known as the modified gauge technique is to formulate an ansatz for the vertex in terms of the fermion propagator that solves the Ward identity for the vertex and preserves the multiplicative renormalizability and gauge covariance of the fermion SD equation [3, 4]. But since the vertex ansatz depends on the fermion propagator, the fermion SD equation is modified to give

$$\frac{\delta V}{\delta S} = \left.\frac{\delta V}{\delta S}\right|_{\Gamma_\mu} + \left.\frac{\delta \Gamma_\mu}{\delta S}\frac{\delta V}{\delta \Gamma_\mu}\right|_{S} = 0. \tag{14}$$

The first term in the middle corresponds to the usual fermion SD equation (Eq. 12) when the second term in the middle vanishes, that is the vertex SD equation (Eq. 13) is satisfied, otherwise there will be extra terms in the fermion SD equation (besides those appearing on the right hand side of Fig. 3).

3. Application to Models of Confinement

Quark confinement in QCD has long been modeled by replacing the gluon propagator appearing in the effective quark propagator SD equation with a $\frac{1}{k^4}$ potential, representing a linearly rising potential in coordinate space [5]. The $\frac{1}{k^4}$ potential has an infrared divergence, which is commonly regulated to give a $\delta^4(k)$ potential [6]. Here we will follow this example and replace the gluon propagator with:

$$\Delta_{\mu\nu}(k) = a g_{\mu\nu}\frac{16\pi^4}{C_f}\delta^4(k), \tag{15}$$

where $C_f = \frac{N^2-1}{2N}$ is the Casimir eigenvalue of the of the fundamental representation of $SU(N)$, and $\sqrt{a}$ is a scale associated with confinement. It is usual at this point to completely truncate the quark-quark-gluon (qqg) vertex by replacing the full qqg vertex Γ_μ, by a bare vertex γ_μ in the quark propagator SD equation, and thus leaving one SD equation to solve [7]. We will alternatively solve the quark propagator SD equation with the full qqg vertex intact. The actual qqg vertex we will use is the modified qqg vertex found by resumming Feynman diagrams to redefine the n-point Green's functions in terms of modified Green's functions [8, 9]. In this case the modified gluon propagator $\hat{\Delta}_{\mu\nu}(k)$ is replaced by Eq. 15, while the full modified qqg vertex, $\hat{\Gamma}_\mu(p,k)$ satisfies a simple Ward identity

$$\hat{\Gamma}_\mu(p,p) = \frac{\partial S^{-1}(p)}{\partial p_\mu}. \tag{16}$$

Using Eq. 16, $\hat{\Gamma}_\mu(p,p)$ can be solved exactly in terms of $S(p)$ without the need to solve the vertex SD equation of Fig. 4. Writing $S(p)$ in terms of scalar functions

$$S(p) = \frac{1}{\alpha \not{p} - \beta(p)}, \tag{17}$$

we find that

$$\hat{\Gamma}_\mu(p,p) = \alpha(p)\gamma_\mu + 2p_\mu \not{p} \frac{\partial \alpha(p)}{\partial p^2} - \{\gamma_\mu, \not{p}\} \frac{\partial \beta(p)}{\partial p^2} \tag{18}$$

satisfies Eq. 16. It is useful to multiply Eq. 18 on the left and right by $S(p)$, so that it becomes

$$S(p)\hat{\Gamma}_\mu(p,p)S(p) = -\frac{\partial S(p)}{\partial p_\mu}, \tag{19}$$

and write $S(p)$ as

$$S(p) = A(p)\not{p} + B(p), \tag{20}$$

giving

$$S(p)\hat{\Gamma}_\mu(p,p)S(p) = -A(p)\gamma_\mu - 2p_\mu \not{p} \frac{\partial A(p)}{\partial p^2} - \{\gamma_\mu, \not{p}\} \frac{\partial B(p)}{\partial p^2}, \tag{21}$$

where $\alpha(p)$ and $\beta(p)$ are related to $A(p)$ and $B(p)$ by

$$\alpha(p) = \frac{A(p)}{p^2 A^2(p) - B^2(p)} \tag{22}$$

$$\beta(p) = \frac{B(p)}{p^2 A^2(p) - B^2(p)} \tag{23}$$

The SD equation for the effective quark propagator is

$$S(p) = S_0(p) + ig^2 C_f S_0(p) \int \frac{d^4k}{16\pi^4} \gamma_\mu S(k) \hat{\Gamma}_\nu(k,p) S(p) \hat{\Delta}^{\mu\nu}(p-k), \tag{24}$$

where

$$S_0(p) = \frac{1}{\not{p}}. \tag{25}$$

Substituting Eq. 15, and Eq. 21 into Eq. 24, we find

$$p^2 \frac{\partial A(p)}{\partial p^2} = \frac{1}{2a} \Big((p^2 - 4a) A(p) - 1 \Big) \tag{26}$$

$$\frac{\partial B(p)}{\partial p^2} = \frac{B(p)}{2a} \tag{27}$$

Eqs. 26, 27, 22, and 23 have been solved in Euclidean space (i.e. $p^2 = -{p_E}^2$) by Burden, Roberts, and Williams [10], who find for the case of chiral symmetry breaking,

$$A(x) = -\frac{1}{2a} \frac{x - 1 + e^{-x}}{x^2}, \tag{28}$$

$$B(x) = \frac{1}{2\sqrt{a}} e^{-x}, \tag{29}$$

$$\alpha_b(x) = \frac{2x(e^{-x} + x - 1)}{x^3 e^{-2x} + 2(e^{-x} + x - 1)^2}, \tag{30}$$

$$\beta(x) = -\sqrt{a} \frac{2x^3}{x^3 e^{-x} + 2e^{x}(e^{-x} + x - 1)^2}, \tag{31}$$

where $x = \frac{p_E{}^2}{2a}$. For the chiral symmetry preserving case $A(x)$ has the same solution as in Eq. 28, while

$$\alpha_s(x) = \frac{x}{e^{-x} + x - 1}, \tag{32}$$

$$\beta(x) = B(x) = 0. \tag{33}$$

$\alpha_s(x)$, and $\alpha_b(x)$ are the solutions of $\alpha(x)$ corresponding to the chiral symmetric and chiral symmetry breaking cases respectively. To determine which solution is prefered, we need to evalute both sets of solutions in Eq. 9, namely we must calculate

$$\Delta V \equiv V_s - V_b, \tag{34}$$

where V_s is the modified CJT effective potential evaluated with the chiral symmetric solution to $S(p)$, and $\Gamma_\mu(p,p)$ (Eq. 28, 32, and 33), while V_b uses the chiral symmetry breaking solution (Eq. 28-31). Defining

$$\tilde{V} \equiv \frac{1}{n_f N a^2} V(S, \Gamma_\mu), \tag{35}$$

where n_f is the number of fermions, we find as in Ref. 10 that,

$$\tilde{V_{0s}} - \tilde{V_{0b}} = -\frac{1}{2\pi^2} \int x dx \; ln\left(1 + \frac{\bar{B}^2}{2x\bar{A}^2}\right) = -.04, \tag{36}$$

where $\bar{A} \equiv aA$, and $\bar{B} \equiv aB$. $V_2(S)$ in Fig. 1b does not contribute to Eq. 34. The second and third terms in the series on the right hand side of Fig. 1a have contributions to Eq. 34 given by,

$$\begin{aligned} \Delta\tilde{V}|_2 = \quad & -\tfrac{1}{\pi^2} \int x dx \Bigg(x(\bar{A} + 2x\bar{A}')(\alpha_s' - \alpha_b') + (2\bar{A} + x\bar{A}')(\alpha_s - \alpha_b) \\ & \qquad - x\beta' B' \Bigg), \end{aligned} \tag{37}$$

and

$$\begin{aligned} \Delta\tilde{V}|_3 = \quad & -\tfrac{2}{\pi^2} \int x dx \Bigg([x\bar{A}^2 - 2x^2\bar{A}'(\bar{A} + x\bar{A}')](\alpha_s\alpha_s' - \alpha_b\alpha_b' + x(\alpha_s'^2 - \alpha_b'^2)) \\ & \qquad + (\bar{A}^2 + x\bar{A}\bar{A}' + x^2\bar{A}'^2)(\alpha_s^2 - \alpha_b^2 - x\bar{\beta}^2) \\ & \qquad + x(\tfrac{1}{2}\alpha_b\bar{A} + x\alpha_b\bar{A}' + x\alpha_b'\bar{A} + 2x^2\alpha_b'\bar{A}')\beta' B' \\ & \qquad - x(\alpha_b^2 + x\alpha_b\alpha_b' + x^2\alpha_b'^2)\bar{B}'^2 + \tfrac{1}{2}x^2\beta'^2 B'^2 \Bigg), \end{aligned} \tag{38}$$

where $\bar{\beta} \equiv a\beta$, and prime denotes partial deifferentiation with respect to x. Eq. 38 has an infrared singularity in $\tilde{V}_{2s}|_3$, which originates from a mismatch in infrared singularities between the left and right sides of Fig. 4. As x approaches zero we find that $\alpha_S = \frac{2}{\epsilon}$, where ϵ is infinitesmal; therefore in the chiral symmetric case, Fig. 4 will have a singularity structure that goes as

$$\frac{b}{\epsilon} = \frac{c}{\epsilon^2} + \frac{d}{\epsilon^3} + ..., \tag{39}$$

where b, c, and d are constants. Presumably the right hand side will sum up to give a power of $\frac{1}{\epsilon}$, which can be achieved by replacing $\delta^4(k)$ in Eq. 15 with $\epsilon\delta^4(k)$ in the limit of small x to obtain a power of $\frac{1}{\epsilon}$ for each term on the right hand side of Fig. 4. But such a resummation is only valid in the infrared regime, and we are interested in performing an integral over all momenta. We notice instead that the infrared singularity in Eq. 38 is isolated to the term involving $\alpha_s^2\bar{A}^2$ (other terms are also infrared singular, but the singularity cancels among them); in fact in graphs involving more than four vertices the singularity is also dominate in the term containing a power of $\alpha_s\bar{A}$. We find that terms invoving $\alpha_s\bar{A}$ and $\alpha_b\bar{A}$ resum into a logarithm for the graphs shown in Fig. 1a (and the graph involving 6 vertices, we have not checked 8 and higher vertex graphs). Replacing the sum of the terms on the right hand sides of Eqs. 37 and 38 involving $\bar{A}(\alpha_s - \alpha_b)$, and $\bar{A}^2({\alpha_s}^2 - {\alpha_b}^2)$ respectively by,

$$-\frac{1}{\pi^2}\int xdx\left(4(\alpha_s - \alpha_b)\bar{A} + ln\left(\frac{1-2\alpha_s\bar{A}}{1-2\alpha_b\bar{A}}\right)\right), \tag{40}$$

and then adding Eqs. 36, 37, and 38, we find that

$$\tilde{V_s} - \tilde{V_b} = .1 > 0. \tag{41}$$

To accurately determine $\tilde{V_s} - \tilde{V_b}$, we need to sum the infinite terms on the right hand side of Eq. 1a for both the symmetric and symmetry breaking cases, which may be tractable, but will not be pursued here. By keeping 3 terms in the series in Fig. 1a, we have demonstrated the possibility that the chiral symmetry breaking solution is the physical solution to Eq. 24 unlike the conclusion reached in Ref. 11.

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Figure Captions

Figure 1. The interaction part of the effective potential.
Figure 2. The gauge particle propagator SD equation.
Figure 3. The fermion propagator SD equation.
Figure 4. The vertex SD equation.

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